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A Maximum-Rank Minimum-Term-Rank Theorem for Matroids

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ABSTRACT

M. Iri has proved that the maximum rank for a pivotal system of matrices (i.e., combivalence class) equals the minimum term rank. Here this and some of Iri's related results are generalized to matroids. These generalizations are presented using a representation of matroids with (0, 1)-matrices. Then, with the aid of matroid basis graphs, these generalizations are restated graph-theoretically. Finally, related results about certain uniform basis graphs are derived.

Let V be a finite set of vectors in some space over a field F. For any maximal independent set $\{x_1, x_2, \dots, x_m\} = X \subset V$, let $\{y_1, y_2, \dots, y_n\} = Y = V - X$. Let $M(X) = [\alpha_{ij}]$ be the $m \times n$ matrix over F, unique up to order, such that

$$y_j = \sum_{i=1}^{m} \alpha_{ij} x_i, \qquad 1 \leqslant j \leqslant n.$$

Following Tucker [8], we call the collection of all such matrices arising from V a *combivalence class* (CC for short). Such collections are often called pivotal systems; for, given any matrix, the combivalence class containing it is just the set of all matrices obtainable from it by the standard pivot exchange procedure of linear programming.

In 1968, Iri [3] gave a long constructive proof of the following theorem: the maximum rank for matrices in a combivalence class equals the minimum term rank; moreover, some matrix in the class has both. Actually, it follows immediately from some of Iri's auxiliary results that one may say slightly more: in a CC *every* matrix with max rank has min term rank, so a matrix has max rank if and only if its rank and term rank are equal. Here, as usual, the rank of a matrix is the dimension of the column space, and the term rank

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is the largest number of non-zero entries such that no two are in the same row or column.

It is well known that finite sets of vectors, or equivalently CC's, form a large but proper class of matroids. Hence it is reasonable to ask if Iri's result generalizes to matroids. We show here that it does. To every matroid there corresponds a collection of $(0, 1)$-matrices which we call its *pseudo-combivalence class* (PC); except for the replacement of CC by PC, the statement of our generalization is the same as Iri's original.

As for proving this generalization, in a sense this has already been done. D. R. Fulkerson, in a private communication with Iri [2], noted that Iri's result is closely related to Theorem 2b in Edmonds and Fulkerson [1]. Using this theorem, which deals with disjoint independent sets of prescribed sizes in arbitrary matroids, Fulkerson outlined a simple proof that max rank equals min term rank for CCs. With a little care, the same argument works for PCs. Below we give a slight variation of that argument, one which uses a somewhat better-known matroid theorem, the union rank equation (1). We also generalize the second part of Iri's theorem, that every matrix with max rank has min term rank. Fulkerson did not discuss this in his correspondence.

Our proof of the generalization is not directly constructive, but we note that (1), also Theorem 2b of [1], have constructive proofs, and this implies a construction for our generalization.

Iri also gave several related results, including some about a concept he called block rank. We will state generalizations for a few of these too. Finally, we translate some of these into the language of matroid basis graphs, where they make for interesting statements and suggest one or two other results about certain uniform basis graphs.

Since first writing this paper we have had several valuable conversations on the subject with Professor Jack Edmonds. He has observed that our generalization can be proved simply using his matroid intersection theorem. Also, he has shown how to state and demonstrate our generalization using purely matroid-theoretic concepts, that is, without any pseudo-combivalent matrices, which are admittedly somewhat of an artifice. We look forward to a paper about this by him soon.

PRELIMINARIES

A *matroid* $\mathbf{M}$ is a finite set of *elements* E and a collection $\mathbf{I}$ of subsets of E, called *independent sets*, such that

(i) $\emptyset \in \mathbf{I}$;

(ii) if $I \in \mathbf{I}$ and $I' \subset I$, then $I' \in \mathbf{I}$;

(iii) if $I, I' \in \mathbf{I}$ and $|I| < |I'|$, then for some $e \in I' - I$, $I + e \in \mathbf{I}$.

Here $|I|$ is the cardinality of I, $I+e$ is shorthand for $I \cup \{e\}$, and $I-I'$ is the set of elements in I but outside I'.

If E is a finite set of vectors in some space, and $\mathbf{I}$ is the collection of subsets *linearly* independent, then $\mathbf{I}$ satisfies (i)–(iii) and we have, as suggested previously, a *vector matroid.*

In any matroid $\mathbf{M}(E)$, the sets of $\mathbf{I}$ maximal by inclusion are called *bases.* By (iii) any two bases B, B' of $\mathbf{M}$ are equicardinal. Moreover, it is not hard to show that for each $e' \in B'-B$ there exists $e \in B-B'$ such that $B-e+e'$ is also a basis. Indeed, this *exchange property* characterizes matroids.

For any $A \subset E$, its rank $\rho(A)$ is the greatest cardinality of any independent subset of A. Given $\mathbf{M}(E)$ and $\mathbf{M}'(E)$, it happens that $\{I \cup I' | I \in \mathbf{I}, I' \in \mathbf{I}\}$ satisfies (i)–(iii), so we get another matroid $\mathbf{M} \vee \mathbf{M}'$: The rank function of this *union*, or *Edmonds sum*, is written $\rho \vee \rho'$. One can show [6] that

$$(\rho \vee \rho')(E) = \min_{A \subset E} \{\rho(A) + \rho'(A) + |E-A|\}. \tag{1}$$

Now suppose $\mathbf{M}(E)$ has basis collection $\mathbf{B}$. For each $B \in \mathbf{B}$ we define $\mathfrak{M}(B)$ to be the (0,1)-matrix with rows B and columns $C = E - B$ such that the (b,c) entry is 1 if and only if $B-b+c \in \mathbf{B}$. We call $\{\mathfrak{M}(B) | B \in \mathbf{B}\}$ the *pseudo-combivalence* class of $\mathbf{M}$. One should not view the columns of $\mathfrak{M}(B)$ as vectors, for in general they do not express the c's as linear combinations of the b's. Thus the only appropriate definition of *rank* for $\mathfrak{M}(B)$ is $\rho(C)$. In particular, if $\mathbf{M}$ is a vector matroid, so that it has both a CC and a PC, then it follows that $\mathfrak{M}(B)$ and $M(B)$ (as defined in the first paragraph) have the same rank. They also have the same term rank, for it is easy to see that $\mathfrak{M}(B)$ is just $M(B)$ with each non-zero entry changed to 1. In light of the last two sentences, our restatement of Iri's theorem for PCs will truly be a generalization of the original statement for CCs.

Note. As suggested by the lack of a relevant vector interpretation for columns, pseudo-combivalence is hardly as useful as combivalence. Surprisingly though, the former has almost as much pivotal structure as the latter does over the field F_2, whence the name. For elaboration and applications, see [5, 9].

Finally, we will use the famous König-Egerváry theorem: the term rank of a matrix is equal to the minimal number of rows and columns needed to cover all the non-zero entries. See, e.g., [7, p. 55].

THEOREM 1. *Let $\mathbf{M}'$ be a matroid and let $\mathfrak{M}$ be its pseudo-combivalence class. Then the maximum rank for matrices in $\mathfrak{M}$ equals the minimum term rank. Moreover, every matrix in $\mathfrak{M}$ with maximum rank also has minimum term rank.*

Proof. The rank of $\mathcal{M}(B)$ is just the maximal size of an independent set disjoint from B. Thus the max rank in $\mathfrak{M}$ is the maximum value of $|I|$ for pairs B, I satisfying

$$B \in \mathbf{B}, \qquad I \in \mathbf{I}, \tag{2}$$

$$B \cap I = \emptyset. \tag{3}$$

Now we claim that every basis of $\mathbf{M} \vee \mathbf{M}$ is of the form $B \cup I$ for B, I as above and I maximal. Every basis is at least of the form $B \cup I_2$ satisfying (2), for given any independent set $I_1 \cup I_2$ in $\mathbf{M} \vee \mathbf{M}$, we have $I_1 \cup I_2 \subset B \cup I_2$ whenever $I_1 \subset B$. Now let $I = I_2 - B$, and (3) is satisfied. Also, $|I|$ must now be maximal, else $B \cup I$ is not a basis. Summing up, if q is the max rank of $\mathfrak{M}$,

$$\rho(E) + q = (\rho \vee \rho)(E). \tag{4}$$

Now let r be the min term rank for $\mathfrak{M}$ and let $r(B)$ be the term rank of $\mathcal{M}(B)$. In light of (1) and (4), to prove that max rank equals min term rank we need only show that

$$\min_{A \subset E} \{2\rho(A) + |E - A|\} = \rho(E) + r.$$

For a given A, let A_1 be a maximal independent subset of A, and set $A_2 = A - A_1$. Let B be a basis containing A_1. Clearly $B \cap A_2 = \emptyset$, so $\mathcal{M}(B)$ breaks up as follows:

$$\begin{array}{r|c|c|} \cline{2-3} A_1 & \mathcal{M}_1 & \mathcal{M}_3 \\ \cline{2-3} B - A_1 & \mathcal{M}_2 & \mathcal{M}_4 = 0 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{C} & \multicolumn{1}{c}{A_2} \end{array} \, . \tag{5}$$

$\mathcal{M}_4$ is a zero matrix, because otherwise, by definition of $\mathcal{M}(B)$, we could exchange some element of A_2 for one of $B - A_1$, and A_1 would not be maximally independent in A. Thus $\mathcal{M}(B)$ can be covered by the rows of A_1 and the columns of C. We have

$$\begin{aligned} 2\rho(A) + |E - A| &= |B| + |A_1| + |C| \\ &\geqslant |B| + r(B) \geqslant |B| + r, \end{aligned} \tag{6}$$

where the first inequality is from the König-Egerváry theorem.

Now pick B such that $r(B)=r$ and cover $\mathfrak{M}(B)$ with a minimal cover of r rows and columns. Labelling these rows A_1 and these columns C, and perhaps rearranging, we again get (5). Let $A=A_1\cup A_2$. We must have $\rho(A)=|A_1|$, for otherwise by axiom (iii) some basis would contain A_1 and some $a\in A_2$, and then by the exchange property $\mathfrak{M}_4$ would not be 0. Thus

$$2\rho(A)+|E-A|=|B|+r=(\rho\vee\rho)(E), \tag{7}$$

and the first part of the theorem is proved. (We repeat that the argument so far is essentially due to Fulkerson.)

Now let A be any set which satisfies (7). Let $\mathfrak{M}(B)$ be any matrix of max rank, and set $A_1=B\cap A$. We claim that once again $\rho(A)=|A_1|$. If so, again we get (5), and $\mathfrak{M}(B)$ has term rank r by (6) and (7).

As for this claim, if $\mathfrak{M}(B)$ has max rank, there is a basis $B\cup I$ of $M\vee M$ satisfying (2) and (3). Temporarily let A be any subset of E. Clearly

$$\rho(A)\geqslant|B\cap A|,\qquad \rho(A)\geqslant|I\cap A|,$$
$$|E-A|\geqslant|(B\cup I)-A|. \tag{8}$$

Thus

$$2\rho(A)+|E-A|\geqslant|(B\cup I)\cap A|+|(B\cup I)-A|$$
$$=|B\cup I|=(\rho\vee\rho)(E).$$

[This is essentially the standard argument to prove the easier inequality of (1).] Hence if A satisfies (7), all the inequalities of (8) must actually be equalities. The first of these equalities proves the claim. ∎

Remark. A careful reading of our proof of Theorem 1 shows that the first part of that theorem and the case $\rho=\rho'$ of (1) are actually equivalent.

As immediate corollaries, we also get generalizations of two other results of Iri, the first of which he proved in the process of obtaining his main result.

Corollary 1. *Let $\mathfrak{M}$ be a matrix in the pseudo-combivalence class of some matroid. If the term rank of $\mathfrak{M}$ is greater than its rank, then some other matrix in the class has greater rank.*

Corollary 2. *With $\mathfrak{M}$ as above, if the term rank of $\mathfrak{M}$ is one greater than its rank, then that term rank is the minimal term rank (= maximal rank) of the pseudo-combivalence class.*

Finally, Iri noted from Corollary 2 that rank and term rank do not have entirely symmetrical properties. Let us elaborate slightly. Consider the pseudo-combivalence class of those matrices actually combivalent over F_2 to the matrix on the left in (9):

$$\boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}}, \quad \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}}. \tag{9}$$

All matrices in this PC, up to order of rows and columns, are of one of these two forms. They all have term rank 2, but those of the form on the right have rank 1. Thus it is not true that a matrix has min term rank if *and only if* it has max rank, or if *and only if* its rank and term rank are equal.

BLOCK RANK

Let B be a basis of the matroid $\mathbf{M}(E)$. We say that (A, C) is a *pivot block* of $\mathfrak{M}(B)$ if $A \subset B$, $C \subset E - B$, and $B - A + C$ is a basis. Under these conditions, one may start with $\mathfrak{M}(B)$ and in $|A| = |C|$ consecutive pivot steps exchange all the elements of A for those of C. If $\mathbf{M}$ is representable by a combivalence class, it also follows that the minor of $M(B)$ indexed by rows A and columns C is non-singular.

Pivot blocks (A, C) and (A', C') are *disjoint* if $A \cap A' = C \cap C' = \emptyset$. Let us define the *n-block rank* of a matroid basis, and likewise of a matrix in a PC, as the maximum value of $\Sigma_{i=1}^{n}|A_i| = \Sigma_{i=1}^{n}|C_i|$, where the (A_i, C_i) are mutually disjoint block pivots, and some may be the null block $(\emptyset, \emptyset)$. This generalizes Iri's notion of the n-block rank of a matrix M, namely, the maximum value of $\Sigma_{i=1}^{n}|A_i|$, where the (A_i, C_i) index disjoint, perhaps null, non-singular minors. All of Iri's results about block rank generalize too. Most of these results were obvious, but in particular we have the non-obvious

THEOREM 2. *Every matrix in a pseudo-combivalence class has the same 2-block rank, namely, the minimum term rank (= maximum rank) for that entire class.*

This theorem follows immediately from a previously published result about matroid basis graphs, so we delay a proof until the next section.

BASIS GRAPHS

If $\mathbf{M}$ is a matroid, the *basis graph* BG(M) is defined to have a vertex for each basis of $\mathbf{M}$ and an edge between two bases if and only if they differ by a single pivot exchange. Thus BB' is an edge if and only if $|B-B'|=|B'-B|=1$. Basis graphs were characterized in [4].

In any graph, a path of shortest length between two vertices is said to be a *geodesic*. $\delta(v,v')$, the *distance* between vertices v,v', is the length of such a geodesic, i.e., the number of edges in it. A geodesic of maximal length among all those with v as an endpoint is called *a radius* from v. (N.B. This is a broader definition of radius than used by some.) A geodesic of maximal length among all geodesics is *a diameter*. Thus a path is a diameter if and only if it is a maximal radius. If v,v' are the endpoints of a diameter, the number $\delta(v,v')$ is *the diameter*. Likewise, we may speak of *the radius* from v.

Given a vertex v, suppose U is a collection of vertices each of which is adjacent to v but no two of which are adjacent to each other. If $|U|$ is maximal among all sets of vertices having this property, the set $U+v$, along with the edges between U and v, is called a *claw* at v, and $|U|$ is the *claw size* at v.

THEOREM 1A. *In the basis graph of a matroid, the minimal claw size equals the diameter. Moreover, every vertex at the end of a diameter has this minimal claw size.*

Proof. We need merely show that the claw size at B is the term rank of $\mathfrak{M}(B)$ and that the radius from B is the rank of $\mathfrak{M}(B)$. As for the first, recall that the (i,j) entry of $\mathfrak{M}(B)$ is 1 if and only if $B-b_i+c_j$ is also a basis. In addition, it is immediate from the definition of basis graph that $B-b_i+c_j$ and $B-b_{i'}+c_{j'}$ are not adjacent in BG(M) if and only if both $i\neq i'$ and $j\neq j'$. Thus the maximal number of vertices adjacent to B but not adjacent to each other is just the maximal number of 1's in $\mathfrak{M}(B)$ with no two in the same row or same column.

As for the second, if B,B' are any vertices in BG(M), then by repeated use of the basis exchange property, one gets that $\delta(B,B')=|B'-B|$. Thus $B'-B$ is an independent set disjoint from B and of size $\delta(B,B')$, so the rank of $\mathfrak{M}(B)$ is at least the radius from B. However, by definition of the rank of $\mathfrak{M}(B)$, there is an independent set I disjoint from B with $|I|$ equal to the rank. If B' is any basis containing I, $\delta(B,B')\geqslant|I|$, so the rank of $\mathfrak{M}(B)$ is at most the radius from B. ∎

We get the following amusing corollary about basis graphs with the property that every vertex is at the end of a diameter.

COROLLARY 3. *Let* **M** *be a matroid and* G *its basis graph. Then every radius is a diameter if and only if at every vertex the radius equals the claw size. When these conditions hold, all claws have the same size.*

Proof of Theorem 2. If (A_i, C_i) is a block pivot for $\mathfrak{M}(B)$ and $B_i = B - A_i + C_i$, then $\delta(B, B_i) = |A_i|$. Moreover, (A_1, C_1) and (A_2, C_2) are disjoint if and only if $\delta(B_1, B_2) = |A_1| + |A_2|$. Thus Theorem 2 is equivalent to the statement that every vertex of BG(**M**) is on some diameter. But the author has proved [5, Theorem 5.6] that given any two vertices B', B'' of a basis graph, there is a radius from B' which passes through B''. So simply set $B'' = B$ and let B' be any vertex on the end of a diameter. ■

Because its statement is in the same spirit as Theorem 1A and Corollary 3, we include Theorem 3 below. However, it requires some additional definitions and remarks first.

Suppose a vertex v has a unique vertex farthest away, that is, all radii from v end up at the same place. We say that v has an *antipode*. If every vertex has an antipode, we say that the graph is *antipodal*.

Let **B** be the basis set of **M**(E). The collection $\mathbf{B}^* = \{E - B \mid B \in \mathbf{B}\}$ satisfies the basis exchange property, and the matroid **M*** determined by **B*** is called the *dual* of **M**. We say **M** is *identically self-dual* if **M** = **M***.

Finally, an element of **M** is a *loop* if it is outside every basis, a *coloop* if it is in every basis. If one simply deletes the loops and coloops from E, and hence the coloops from the bases, one obtains a new matroid **M**$'$ with just as many bases as before. Indeed, the graphs BG(**M**) and BG(**M**$'$) are isomorphic. Thus, as far as bases are concerned, loops and coloops can be ignored.

THEOREM 3. *A matroid without loops or coloops is identically self-dual if and only if its basis graph is antipodal. When these conditions hold, the distance between every vertex and its antipode is the same, namely the diameter.*

Proof. If **M**(E) is identically self-dual and B is a basis, then $|B - (E - B)| = |B| > |B - B'|$, where B' is any basis other than $E - B$. Thus $E - B$ is the unique basis farthest from B, and the radius from B, being independent of B, is the diameter.

Conversely, let B be any basis and B' its antipode. We claim that $B' = E - B$. For suppose $b \in B \cap B'$. Since **M** has no coloops, there exists a basis B_1 disjoint from b. By the result from [5] quoted in the proof of Theorem 2, there is a radius from B passing through B_1. Let us call the other end B_2. B_2 must also be disjoint from b, else no geodesic from B to B_2 would pass through B_1. Hence $B_2 \neq B'$, contradicting the assumption that BG(**M**) is antipodal. On the other hand, suppose there exists $e \in (E-B)-B'$. Since **M** has no loops, there must be a basis B_3 containing e. By another radius argument we again get the impossible conclusion that B has another antipode. This proves the claim, and consequently **M** is identically self-dual. ■

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